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A modified form of the κ–ε model for turbulent flows of an incompressible fluid in porous media

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Abstract

This work extends the recently published work of Antohe and Lage on the development of a macroscopic two-equation turbulence model for an incompressible flow in porous media. The difference occurs in approximating the Forchheimer term in the time-averaged momentum equation. Unlike the Antohe and Lage's work, where only the linear terms of the expansion are kept, in the presently proposed model we include the second-order correlation term. This inclusion gives rise to an extra term in the transport momentum equation which, in turn, gives rise to additional terms in the transport equations for the turbulent kinetic energy and the dissipation rate. The additional higher order terms produce correlation coefficients that are used to absorb any departure from the clear flow when expressing the two-equation turbulence model for incompressible flow in a porous medium. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In a recent paper, Antohe and Lage [1] presented a detailed derivation of a two-equation model which was proposed for the purpose of describing a nonsteady turbulent flow of an incompressible fluid through a porous medium. By time-averaging the general macroscopic transport equations, including all terms (Nield and Bejan [2]), they extended the theoretical basis for the Reynolds average transport equations for clear flow [3,4] and derived the two-equation turbulence model for incompressible flow in a porous medium. In their derivation, to complete the time-averaging process of the momentum equation, they expanded the Forchheimmer term and kept only the terms which are linear in the fluctuating velocities. Since most of the statistical properties of turbulence are in the second-order correlation terms, neglecting the higher order ones in the fluctuating velocities may hide some important effects of the Forchheimmer term.

This paper describes the development of a modified form of [1] which includes the effect of the second-order term in the approximation of the Forchheimer term. The philosophical basis for this idea resides in the expectation that suppression of modeling in the hierarchy of moment equations to the second level incorporates more of the mechanics of turbulence and thus leads to a more accurate formulation. It can be argued that if models of the first order lead to a

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reasonable description, as suggested in Ref. [1], it might be expected that higher order approximation might lead to a better description of the turbulence effects of the Forchheimer term as well as to more accurate solution of the mean momentum and mean temperature equations in flows through porous media.

2. Formulation of model equations

The ensemble average of the volume-averaged equations governing the transport of mass, momentum and energy in a rigid, isotropic and fixed porous matrix, for a constant-property Newtonian fluid can be written as [1]:

$$\frac{\partial U_i}{\partial x_i}=0 \tag{1}$$

$$\begin{aligned}\frac{\partial U_i}{\partial t}+\frac{\partial\left(U_jU_i+\overline{\grave{u}_j\grave{u}_i}\right)}{\partial x_j}&=-\frac{1}{\rho_f}\frac{\partial P}{\partial x_i}-\phi\frac{\nu}{K}U_i\\&\quad-\phi^2\frac{c_F}{K^{1/2}}\overline{\left[(U_j+\grave{u}_j)(U_j+\grave{u}_j)\right]^{1/2}(U_i+\grave{u}_i)}\\&\quad+\nu J\frac{\partial^2U_i}{\partial x_j\partial x_j}-\delta_{i3}\left[1-\beta(\Theta-T_o)\right]g\end{aligned} \tag{2}$$

$$(\rho c)_e\frac{\partial\Theta}{\partial t}+\phi(\rho c)_f\frac{\partial\left(U_j\Theta+\overline{\grave{u}_j\grave{\theta}}\right)}{\partial x_j}=k_e\frac{\partial^2\Theta}{\partial x_j\partial x_j} \tag{3}$$

where ($\grave{u}_i$, $\grave{p}$, $\grave{\theta}$) represent the fluctuating quantities and (U_i, P, Θ) represent the ensemble averages. Further details of this Reynolds averaging procedure for clear flow are available in Refs. [3] and [4].

The third term on the right-hand side of Eq. (2), the Forchheimer term, may be expanded as

$$\begin{aligned}&\overline{\left[(U_j+\grave{u}_j)(U_j+\grave{u}_j)\right]^{1/2}(U_i+\grave{u}_i)}\\&\quad=\overline{\left(U_jU_j+2U_j\grave{u}_j+\grave{u}_j\grave{u}_j\right)^{1/2}(U_i+\grave{u}_i)}\\&\quad=\overline{\left(U_jU_j+2U_j\grave{u}_j\right)^{1/2}(U_i+\grave{u}_i)}\end{aligned} \tag{4}$$

where we assume that $U_jU_j\gg\grave{u}_j\grave{u}_j$. Next, using the Binomial Theorem, the power series expansion of the quantity $(U_jU_j+2U_j\grave{u}_j)^{1/2}$ is

$$\begin{aligned}&\left(U_jU_j+2U_j\grave{u}_j\right)^{1/2}\\&\quad=\left(U_jU_j\right)^{1/2}+\frac{1}{2}\frac{2U_j\grave{u}_j}{\left(U_jU_j\right)^{1/2}}-\frac{1}{8}\frac{\left(2U_j\grave{u}_j\right)^2}{\left(U_jU_j\right)^{3/2}}\\&\qquad+\frac{1}{16}\frac{\left(2U_j\grave{u}_j\right)^3}{\left(U_jU_j\right)^{5/2}}+\cdots\\&\quad=\left(U_jU_j\right)^{1/2}\left[1+\frac{U_j\grave{u}_j}{U_jU_j}-\frac{1}{2}\left(\frac{U_j\grave{u}_j}{U_jU_j}\right)^2+\frac{1}{2}\left(\frac{U_j\grave{u}_j}{U_jU_j}\right)^3+\cdots\right]\end{aligned} \tag{5}$$

Neglecting the higher order terms in $(U_j\grave{u}_j/U_jU_j)$, Eq. (5) yields

$$\left(U_jU_j+2U_j\grave{u}_j\right)^{1/2}\approx\left(U_jU_j\right)^{1/2}+\frac{U_j\grave{u}_j}{\left(U_jU_j\right)^{1/2}} \tag{6}$$

Substitution of Eq. (6), along with the averaging rules [3], into Eq. (4) gives

$$\begin{aligned}&\overline{\left[(U_j+\grave{u}_j)(U_j+\grave{u}_j)\right]^{1/2}(U_i+\grave{u}_i)}\\&\quad\approx\overline{\left(\left(U_jU_j\right)^{1/2}+\frac{U_j\grave{u}_j}{\left(U_jU_j\right)^{1/2}}\right)(U_i+\grave{u}_i)}\\&\quad\approx\overline{\left(U_jU_j\right)^{1/2}U_i+\left(U_jU_j\right)^{1/2}\grave{u}_i+\frac{U_jU_i\grave{u}_j}{\left(U_jU_j\right)^{1/2}}+\frac{U_j\grave{u}_j\grave{u}_i}{\left(U_jU_j\right)^{1/2}}}\\&\quad\approx\left(U_jU_j\right)^{1/2}U_i+\frac{U_j}{\left(U_jU_j\right)^{1/2}}\overline{\grave{u}_j\grave{u}_i}\end{aligned} \tag{7}$$

Here is where the present derivation deviates from that of [1]. In that derivation, the second term was neglected and only the first term was retained. For the reasons given in Section 1, the present derivation retains the second term in approximating the Forchheimer effect.

Substituting Eq. (7) into Eq. (2), the averaged momentum equation becomes

$$\begin{aligned}\frac{\partial U_i}{\partial t}+\frac{\partial\left(U_jU_i+\overline{\grave{u}_j\grave{u}_i}\right)}{\partial x_j}&=-\frac{1}{\rho_f}\frac{\partial P}{\partial x_i}\\&\quad-\phi\frac{\nu}{K}U_i-\phi^2\frac{c_F}{K^{1/2}}\left[\left(U_jU_j\right)^{1/2}U_i+\frac{U_j}{\left(U_jU_j\right)^{1/2}}\overline{\grave{u}_j\grave{u}_i}\right]\\&\quad+\nu J\frac{\partial^2U_i}{\partial x_j\partial x_j}-\delta_{i3}\left[1-\beta(\Theta-T_o)\right]g\end{aligned}$$

The energy equation (3) and the transport momentum equation (8) can now be written in their most general form as

$$(\rho c)_e \frac{\partial \Theta}{\partial t} + \phi(\rho c)_f U_j \frac{\partial \Theta}{\partial x_j} = k_e \frac{\partial^2 \Theta}{\partial x_j \partial x_j} - \phi(\rho c)_f \frac{\partial \left(\overline{\grave{u}_j \grave{\theta}}\right)}{\partial x_j} \tag{9}$$

$$\begin{aligned} \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} &= -\frac{1}{\rho_f}\frac{\partial P}{\partial x_i} + \nu J \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{\partial \left(\overline{\grave{u}_j \grave{u}_i}\right)}{\partial x_j} \\ &\quad - \delta_{i3}\left[1 - \beta(\Theta - T_o)\right] g - \phi \frac{\nu}{K} U_i \\ &\quad - \phi^2 \frac{c_F}{K^{1/2}} \left[\left(U_j U_j\right)^{1/2} U_i + \frac{U_j}{\left(U_j U_j\right)^{1/2}} \overline{\grave{u}_j \grave{u}_i} \right] \end{aligned} \tag{10}$$

It should be noted that, except for the last term on the right-hand side of Eq. (10), Eqs. (9) and (10) are identical to the corresponding terms in Ref. [1].

Classical closure of the transport momentum and the energy equations expresses the Reynolds flux, $\overline{\grave{u}_i \grave{u}_j}$, and the turbulent flux of the temperatures, $\overline{\grave{\theta} \grave{u}_j}$, as [3,4]:

$$-\overline{\grave{u}_i \grave{u}_j} = \nu_t S_{ij} - \frac{2}{3} \kappa \delta_{ij} \tag{11a}$$

and

$$-\overline{\grave{\theta} \grave{u}_i} = \frac{\nu_t}{\sigma_t} \frac{\partial \Theta}{\partial x_i} \tag{11b}$$

where S_{ij} is the mean strain-rate tensor, $\kappa = \frac{1}{2}\overline{\grave{u}_i \grave{u}_i}$ is the turbulent kinetic energy, ν_t is the turbulent kinematics coefficient and σ_t is the turbulent Prandtl number.

For the κ–ε model, with κ and ε treated as principle dependent variables, the following model for ν_t is proposed:

$$\nu_t = C_\mu \frac{\kappa^2}{\varepsilon} \tag{12}$$

where C_μ is an empirical coefficient. This concept introduces additional physics (local values of κ and ε) in the formulation. Thus, implementation of such a model calls for equations for ε and κ. Formal derivations of partial differential equations for the turbulent kinetic energy and the mean dissipation rate are presented in the following two subsections.

2.1. Model equation for κ

The transport equation for the turbulence kinetic energy can be obtained by a simple contraction of the exact transport equation for the correlation $\overline{\grave{u}_i \grave{u}_j}$. A straightforward approach to find the transport equation for $\overline{\grave{u}_i \grave{u}_j}$ is to simply form the second moment equation [4,5]

$$\overline{\grave{u}_i \aleph(u_j) + \grave{u}_j \aleph(u_i)} = 0 \tag{13}$$

where $\aleph(u_i)$ is the instantaneous volume-averaged momentum equation in operator form, defined as

$$\begin{aligned} \aleph(u_i) = {} & \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho_f}\frac{\partial p}{\partial x_i} - \nu J \frac{\partial^2 u_i}{\partial x_j \partial x_j} \\ & + \delta_{i3}\left[1 - \beta(\theta - T_o)\right] g + \phi \frac{\nu}{K} u_i \\ & + \phi^2 \frac{c_F}{K^{1/2}} (u_j u_j)^{1/2} u_i \end{aligned} \tag{14}$$

More explicitly, the Reynolds-stress transport Eq. (13) is given by

$$\begin{aligned} \frac{\partial \overline{\grave{u}_i \grave{u}_j}}{\partial t} + U_k \frac{\partial \overline{\grave{u}_i \grave{u}_j}}{\partial x_k} = {} & -\overbrace{\overline{\grave{u}_i \grave{u}_k} \frac{\partial U_j}{\partial x_k} - \overline{\grave{u}_j \grave{u}_k} \frac{\partial U_i}{\partial x_k}}^{P_{ij}} + \overbrace{\beta g\left[\delta_{j3} \overline{\grave{u}_i \grave{\theta}} + \delta_{i3} \overline{\grave{u}_j \grave{\theta}}\right]}^{G_{ij}} - \overbrace{\frac{2\phi\nu}{K} \overline{\grave{u}_i \grave{u}_j}}^{Da_{ij}} \\ & - \overbrace{\frac{\phi^2 c_F}{(U_m U_m)^{1/2} K^{1/2}} \left[2 \overline{\grave{u}_i \grave{u}_j} (U_m U_m) + \overline{\grave{u}_j \grave{u}_k} (U_i U_k) + \overline{\grave{u}_i \grave{u}_k} \left(U_j U_k\right)\right]}^{\mathscr{F}_{1ij}} + \overbrace{\frac{\partial}{\partial x_k}\left[\nu J \frac{\partial \overline{\grave{u}_i \grave{u}_j}}{\partial x_k} - \frac{1}{\rho_f} \delta_{jk} \overline{\grave{u}_i \grave{p}} - \frac{1}{\rho_f} \delta_{ik} \overline{\grave{u}_j \grave{p}} - \overline{\grave{u}_i \grave{u}_k \grave{u}_j}\right]}^{\mathscr{D}_{ij}} \\ & + \overbrace{\frac{\overline{\grave{p}}}{\rho_f} \left(\frac{\partial \grave{u}_i}{\partial x_j} + \frac{\partial \grave{u}_j}{\partial x_i}\right)}^{\Pi_{ij}} - \overbrace{2 \frac{\phi^2 c_F U_k}{(U_m U_m)^{1/2} K^{1/2}} \overline{\grave{u}_i \grave{u}_j \grave{u}_k}}^{\mathscr{F}o_{ij}} - \overbrace{2\nu J \overline{\frac{\partial \grave{u}_i}{\partial x_k} \frac{\partial \grave{u}_j}{\partial x_k}}}^{\varepsilon_{ij}} \end{aligned} \tag{15}$$

Thus, contraction of Eq. (15) yields the following model equation for κ

$$\frac{\partial\overline{\grave{u}_i\grave{u}_i}}{\partial t}+U_k\frac{\partial\overline{\grave{u}_i\grave{u}_i}}{\partial x_k} = \overbrace{-2\overline{\grave{u}_i\grave{u}_k}\frac{\partial U_i}{\partial x_k}}^{P_{ii}}+\overbrace{2\delta_{i3}\beta g\overline{\grave{u}_i\grave{\theta}}}^{G_{ii}}-\overbrace{\frac{2\phi\nu}{K}\overline{\grave{u}_i\grave{u}_i}}^{Da_{ii}}$$
$$-\overbrace{\frac{\phi^2 c_F}{(U_kU_k)^{1/2}K^{1/2}}\left[2\overline{\grave{u}_i\grave{u}_i}(U_kU_k)+2\overline{\grave{u}_i\grave{u}_k}(U_iU_k)\right]}^{\mathscr{F}_{ii}}$$
$$+\overbrace{\frac{\partial}{\partial x_k}\left[\nu J\frac{\partial\overline{\grave{u}_i\grave{u}_i}}{\partial x_k}-2\frac{1}{\rho_f}\delta_{ik}\overline{\grave{u}_i\grave{p}}-\overline{\grave{u}_i\grave{u}_k\grave{u}_i}\right]}^{\mathscr{D}_{ii}}$$
$$-\overbrace{2\nu J\overline{\frac{\partial\grave{u}_i}{\partial x_k}\frac{\partial\grave{u}_i}{\partial x_k}}}^{\varepsilon_{ii}}-\overbrace{2\frac{\phi^2c_FU_k}{(U_kU_k)^{1/2}K^{1/2}}\overline{\grave{u}_i\grave{u}_i\grave{u}_k}}^{\mathscr{F}1_{ii}} \tag{16}$$

The terms above have been grouped, following well-established practices, so as best to allow a physical interpretation of the processes. A short-hand symbol for the process in question, which we shall use later to simplify the equation, appears over each group of terms. The first four processes can be treated exactly while the last three groups of terms contain unknown correlations.

Comparing Eq. (16) with Eq. (17) of Ref. [1], we can see that Eq. (16), obtained as the result of the second-order approximation of the Forchheimer term, contains an extra term which is of a third-order level. Thus, unlike the model equation presented in Ref. [1], in the present model the third-order correlation appears twice. To close the system, both terms containing the triple-velocity correlations, along with the other correlation terms, which appeared in the equation, must be modeled.

The terms

$$2\frac{1}{\rho_f}\delta_{ik}\overline{\grave{u}_i\grave{p}}+\overline{\grave{u}_i\grave{u}_i\grave{u}_k} \tag{17}$$

in $\mathscr{D}_{ii}$, after being differentiated, can be interpreted as representing a diffusive effect [3,4,6]. Thus, by analogy with the model for the turbulent flux of a scalar, they can be described by a gradient model as

$$\frac{1}{\rho_f}\delta_{ik}\overline{\grave{u}_i\grave{p}}+\frac{1}{2}\overline{\grave{u}_i\grave{u}_i\grave{u}_k}=-\frac{\nu_t}{\sigma_\kappa}\frac{\partial\kappa}{\partial x_k} \tag{18}$$

where σ_κ is a closure coefficient.

In the Forchheimer term, the third-order correlation does not appear differentiated, but can be loosely interpreted as describing the mean transport of the turbulent kinetic energy in the kth coordinate direction. To close the equation, one must express this term as a function of the second-order quantities. For flow without porous media, much discussions of the representation of the triple-velocity correlation were presented in Refs. [4,5]. The three most common models (Hanjalic and Launder [7], Daly and Harlow [8], and Mellor and Herring [9]) used in conjunction with the second-moment closure for this correlation fall into the following general form [5]

$$\overline{\grave{u}_i\grave{u}_j\grave{u}_k}=d_{ijklmn}(\tau,\kappa,\varepsilon)\frac{\partial\tau_{lm}}{\partial x_n} \tag{19}$$

where d_{ijklmn} is shown to be a function of the Reynolds stress as well as the turbulence velocity and length scale. In the presently proposed model (though the model of Daly and Harlow [8] is simple), the expression used by Hanjalic and Launder [7] will be employed. In the derivation of the model equation for the triple-velocity correlation, they [7] modeled the transport equation for the third-order moment, and then extracted an algebraic expression from the modeled equation by neglecting the transport terms and assuming a Gaussian relation to link the fourth-order correlation to the second-order correlation. Their invariant expression for $\overline{\grave{u}_k\grave{u}_i\grave{u}_j}$ is

$$\overline{\grave{u}_k\grave{u}_i\grave{u}_j}=C_t\frac{\kappa}{\varepsilon}\left(\overline{\grave{u}_k\grave{u}_l}\frac{\partial\overline{\grave{u}_i\grave{u}_j}}{\partial x_l}+\overline{\grave{u}_j\grave{u}_l}\frac{\partial\overline{\grave{u}_i\grave{u}_k}}{\partial x_l}+\overline{\grave{u}_i\grave{u}_l}\frac{\partial\overline{\grave{u}_k\grave{u}_j}}{\partial x_l}\right) \tag{20}$$

where C_t is the diffusion constant.

Using the triple correlation model as formulated above, Eq. (16) becomes

$$\frac{\partial\kappa}{\partial t}+U_k\frac{\partial\kappa}{\partial x_k}=\overbrace{-\overline{\grave{u}_i\grave{u}_k}\frac{\partial U_i}{\partial x_k}}^{P_{ii}}+\overbrace{\delta_{i3}\beta g\overline{\grave{u}_i\grave{\theta}}}^{G_{ii}}$$
$$+\overbrace{\frac{\partial}{\partial x_k}\left[\nu J\frac{\partial\kappa}{\partial x_k}+\frac{\nu_t}{\sigma_\kappa}\frac{\partial\kappa}{\partial x_k}\right]}^{\mathscr{D}_{ii}}$$
$$-\overbrace{\frac{2\phi\nu}{K}\kappa}^{Da_{ii}}-\overbrace{\frac{\phi^2c_F}{(U_kU_k)^{1/2}K^{1/2}}\left[2\kappa(U_kU_k)+\overline{\grave{u}_i\grave{u}_k}(U_iU_k)\right]}^{\mathscr{F}_{ii}}$$
$$-\overbrace{\frac{\phi^2c_FU_k}{(U_kU_k)^{1/2}K^{1/2}}C_t\frac{\kappa}{\varepsilon}\left(\overline{\grave{u}_k\grave{u}_l}\frac{\partial\kappa}{\partial x_l}+2\overline{\grave{u}_i\grave{u}_l}\frac{\partial\overline{\grave{u}_k\grave{u}_i}}{\partial x_l}\right)}^{\mathscr{F}1_{ii}}-\overbrace{\frac{J\varepsilon}{2}}^{\varepsilon_{ii}} \tag{21}$$

where

$$\varepsilon = 2\nu \overline{\frac{\partial \grave{u}_i}{\partial x_j}\frac{\partial \grave{u}_i}{\partial x_j}} \tag{22}$$

which appears as sink term, is the dissipation rate of the turbulent kinetic energy. Comparing Eq. (21) to Eq. (19) of Antohe and Lage [1], it is observed that the neglect of the second-order correlation as done in Ref. [1] is equivalent to the neglect of term $\mathscr{F}\imath_{ii}$ in Eq. (21). It is possible to show, via scaling argument, that this term predominates over the term $\mathscr{F}_{ii}$ of Eq. (21) when $\kappa \gg \varepsilon$. Therefore, the model presented here is expected to be more accurate than the model created by Antohe and Lage [1], when the turbulence intensity of the flow is very high.

When applying this turbulence model, the distribution of ε over the flow field must be determined. This is usually achieved, as will be shown in the next section, by solving a transport equation for ε.

2.2. Modeling the dissipation rate

There has been some discussion in the literature (Hanjalic and Launder [6] for clear flow, and Antohe and Lage [1] for porous media flow) as to which form of the ε model is more appropriate. Using the most common approach, the exact equation for ε is derived by taking the following moment of the macroscopic equation

$$2\nu \overline{\frac{\partial \grave{u}_i}{\partial x_j}\frac{\partial}{\partial x_j}\big(\aleph(u_i)\big)} = 0 \tag{23}$$

where $\aleph(u_i)$ is the nonlinear operator defined by Eq. (14). The result may be expressed as

$$\begin{aligned}
\frac{\partial \varepsilon}{\partial t} + U_k\frac{\partial \varepsilon}{\partial x_k} = & -4\nu \overbrace{\left[\overline{\frac{\partial \grave{u}_k}{\partial x_j}\frac{\partial \grave{u}_k}{\partial x_i}} + \overline{\frac{\partial \grave{u}_i}{\partial x_k}\frac{\partial \grave{u}_j}{\partial x_k}}\right]}^{\text{(i)}}\frac{\partial U_i}{\partial x_j} \\
& - 2\nu \overbrace{\frac{\partial}{\partial x_k}\left\{\left[\overline{\grave{u}_k\frac{\partial \grave{u}_i}{\partial x_j}\frac{\partial \grave{u}_i}{\partial x_j}}\right] + \frac{2}{\rho_f}\left[\overline{\frac{\partial \grave{u}_k}{\partial x_j}\frac{\partial \grave{p}}{\partial x_j}}\right]\right\}}^{\text{(iii)}} \\
& - 4\nu \overbrace{\left[\overline{\frac{\partial \grave{u}_i}{\partial x_j}\frac{\partial \grave{u}_i}{\partial x_k}\frac{\partial \grave{u}_k}{\partial x_j}} + \nu J\overline{\frac{\partial^2 \grave{u}_i}{\partial x_k \partial x_j}\frac{\partial^2 \grave{u}_i}{\partial x_k \partial x_j}}\right]}^{\text{(ii)}} \\
& + \nu J\frac{\partial^2 \varepsilon}{\partial x_k \partial x_k} - 4\nu \overbrace{\left[\overline{\grave{u}_k\frac{\partial \grave{u}_i}{\partial x_j}}\right]}^{\text{(iv)}}\frac{\partial^2 U_i}{\partial x_j x_k} + 4\nu\beta g\delta_{i3}\overbrace{\overline{\frac{\partial \grave{u}_i}{\partial x_j}\frac{\partial \grave{\theta}}{\partial x_j}}}^{\text{(v)}} \\
& - 2\frac{\phi\nu}{K}\varepsilon - \frac{2\phi^2 c_F}{K^{1/2}}\left\{(U_m U_m)^{1/2}\frac{\varepsilon}{2}\right. \\
& + \frac{U_k}{(U_m U_m)^{1/2}}\overbrace{\left[\overline{\grave{u}_k\frac{\partial \grave{u}_i}{\partial x_j}\frac{\partial \grave{u}_i}{\partial x_j}} + \overline{\grave{u}_i\frac{\partial \grave{u}_k}{\partial x_j}\frac{\partial \grave{u}_i}{\partial x_j}} + U_i\overline{\frac{\partial \grave{u}_k}{\partial x_j}\frac{\partial \grave{u}_i}{\partial x_j}}\right]}^{\text{(vi)}} \\
& + \nu\frac{\partial}{\partial x_j}(U_m U_m)^{1/2}\overbrace{\left[\overline{\grave{u}_i\frac{\partial \grave{u}_i}{\partial x_j}}\right]}^{\text{(vii)}} + \nu\frac{\partial}{\partial x_j}\left(\frac{U_k}{(U_m U_m)^{1/2}}\right) \\
& \left.\times \overbrace{\left[\overline{\grave{u}_k\grave{u}_i\frac{\partial \grave{u}_i}{\partial x_j}}\right]}^{\text{(viii)}} + \nu\frac{\partial}{\partial x_j}\left(\frac{U_i U_k}{(U_m U_m)^{1/2}}\right)\overbrace{\left[\overline{\grave{u}_k\frac{\partial \grave{u}_i}{\partial x_j}}\right]}^{\text{(ix)}}\right\}
\end{aligned} \tag{24}$$

This equation involves several new unknown double and triple correlations of fluctuating velocity, pressure and velocity gradients, which are usually impossible to measure with any degree of accuracy. Even for clear flow, there is presently little hope of finding reliable guidance from experimentalists regarding suitable closure approximation. Thus, model assumptions have to be introduced in formulating a usable ε equation.

Models for the first four group of terms, which correspond to the clear flow case, are considered first. To approximate these terms, we adopt the clear flow model and absorb any departure from the clear flow in the modeling of the Forchheimer term. In summary (for the derivations of these terms see Refs. [4,6,10]) the closure approximations of these terms are:

$$4\nu\left[\overline{\frac{\partial \grave{u}_k}{\partial x_j}\frac{\partial \grave{u}_k}{\partial x_i}} + \overline{\frac{\partial \grave{u}_i}{\partial x_k}\frac{\partial \grave{u}_j}{\partial x_k}}\right] = C_{\varepsilon 1}\frac{\varepsilon}{\kappa}\overline{\grave{u}_i\grave{u}_j} + \tilde{C}_{\varepsilon 1}\delta_{ij} \tag{25}$$

$$4\left[\nu\overline{\frac{\partial \grave{u}_i}{\partial x_j}\frac{\partial \grave{u}_i}{\partial x_k}\frac{\partial \grave{u}_k}{\partial x_j}} + \nu^2 J\overline{\frac{\partial^2 \grave{u}_i}{\partial x_k \partial x_j}\frac{\partial^2 \grave{u}_i}{\partial x_k \partial x_j}}\right] = C_{\varepsilon 2}\frac{\varepsilon^2}{\kappa} \tag{26}$$

$$2\nu\left[\overline{\grave{u}_k\frac{\partial \grave{u}_i}{\partial x_j}\frac{\partial \grave{u}_i}{\partial x_j}} + \frac{2}{\rho_f}\overline{\frac{\partial \grave{u}_k}{\partial x_j}\frac{\partial \grave{p}}{\partial x_j}}\right] = -\frac{C_{\varepsilon 3}\kappa}{\varepsilon}\overline{\grave{u}_k\grave{u}_j}\frac{\partial \varepsilon}{\partial x_j} \tag{27}$$

$$\nu J\frac{\partial^2 \varepsilon}{\partial x_k \partial x_k} - 4\nu\left[\overline{\grave{u}_k\frac{\partial \grave{u}_i}{\partial x_j}}\right]\frac{\partial^2 U_i}{\partial x_j x_k} \approx 0 \tag{28}$$

$$4\nu\beta\delta_{i3}\overline{\frac{\partial\grave{u}_i}{\partial x_j}\frac{\partial\grave{\theta}}{\partial x_j}}=0 \tag{29}$$

The Forchheimer term contains a number of correlations of turbulent quantities for whose determination a closed path must first be prescribed. These unknown correlations must be approximated by expressions containing mean velocity gradients, Reynolds stresses and ε. The following paragraphs describe the assumptions used to provide reasonable closing approximations of these correlations.

The first term in (vi) represents the turbulent transport of the dissipation. The standard approximation made to represent turbulent transport of scalar quantities in a turbulent flow is that of general gradient-diffusion hypothesis [4]. Employing this hypothesis on the first term and the assumption of local isotropy to the remaining terms, it can be shown that

$$(\text{vi})=\frac{U_k}{(U_mU_m)^{1/2}}\left[-\frac{4}{3}\frac{C_{\varepsilon4}\kappa}{\varepsilon}\overline{\grave{u}_k\grave{u}_l}\frac{\partial\varepsilon}{\partial x_l}+\frac{1}{3}U_i\varepsilon\delta_{ik}\right] \tag{30}$$

The remaining group of terms, (vii), (viii) and (ix), can be simplified to

$$(\text{vii})=\overline{\grave{u}_i\frac{\partial\grave{u}_i}{\partial x_j}}=\frac{1}{2}\frac{\partial\overline{\grave{u}_i\grave{u}_i}}{\partial x_j}=\frac{1}{2}\frac{\partial\kappa}{\partial x_j} \tag{31}$$

where κ is the turbulence kinetic energy, and

$$(\text{viii})+(\text{ix})=\frac{\nu}{2}\frac{\partial}{\partial x_j}\left(\frac{U_kU_i}{(U_mU_m)^{1/2}}\right)\frac{\partial\tau_{ik}}{\partial x_j} \tag{32}$$

Substituting Eqs. (25)–(32) into Eq. (24) one gets

$$\begin{aligned}\frac{\partial\varepsilon}{\partial t}+U_k\frac{\partial\varepsilon}{\partial x_k}&=-\left(C_{\varepsilon1}\frac{\varepsilon}{\kappa}\overline{\tilde{u}_i\tilde{u}_j}+\tilde{C}_{\varepsilon1}\delta_{ij}\right)\frac{\partial U_i}{\partial x_j}-C_{\varepsilon2}\frac{\varepsilon^2}{\kappa}\\&+C_{\varepsilon3}\frac{\partial}{\partial x_k}\frac{\kappa}{\varepsilon}\overline{\tilde{u}_k\tilde{u}_l}\frac{\partial\varepsilon}{\partial x_l})-\frac{2\phi^2c_F}{K^{1/2}}\left\{\frac{1}{2}(U_kU_k)^{1/2}\varepsilon\right.\\&+\frac{U_k}{(U_mU_m)^{1/2}}\left[-\frac{4}{3}C_{\varepsilon4}\frac{\kappa}{\varepsilon}\overline{\tilde{u}_k\tilde{u}_l}\frac{\partial\varepsilon}{\partial x_l}+\frac{U_i}{3}\delta_{ik}\varepsilon\right]\\&+\frac{\nu}{2}\left[\frac{\partial}{\partial x_j}(U_mU_m)^{1/2}\right]\frac{\partial\kappa}{\partial x_j}\\&\left.+\frac{\nu}{2}\frac{\partial}{\partial x_j}\left(\frac{U_kU_i}{(U_mU_m)^{1/2}}\right)\frac{\partial\tau_{ik}}{\partial x_j}\right\}-2\frac{\phi\nu}{K}\varepsilon\end{aligned} \tag{33}$$

Rearranging the above equation, the final form of the dissipation rate equation becomes

$$\begin{aligned}\frac{\partial\varepsilon}{\partial t}+U_k\frac{\partial\varepsilon}{\partial x_k}&=-C_{\varepsilon1}\frac{\varepsilon}{\kappa}\overline{\grave{u}_i\grave{u}_j}\frac{\partial U_i}{\partial x_j}-C_{\varepsilon2}\frac{\varepsilon^2}{\kappa}\\&+C_{\varepsilon3}\frac{\partial}{\partial x_k}\left(\frac{\kappa}{\varepsilon}\overline{\grave{u}_k\grave{u}_l}\frac{\partial\varepsilon}{\partial x_l}\right)\\&-\frac{2\phi^2c_F}{K^{1/2}}\left\{\frac{5}{6}(U_kU_k)^{1/2}\varepsilon\right.\\&+\frac{\nu}{2}\left[\frac{\partial}{\partial x_j}(U_mU_m)^{1/2}\right]\frac{\partial\kappa}{\partial x_j}\\&\left.+\frac{\nu}{2}\frac{\partial}{\partial x_j}\left(\frac{U_kU_i}{(U_mU_m)^{1/2}}\right)\frac{\partial\tau_{ik}}{\partial x_j}\right\}\\&+\frac{8}{3}\frac{\phi^2c_F}{K^{1/2}}\frac{U_k}{(U_mU_m)^{1/2}}\left[C_{\varepsilon4}\frac{\kappa}{\varepsilon}\overline{\grave{u}_k\grave{u}_l}\frac{\partial\varepsilon}{\partial x_l}\right]-2\frac{\phi\nu}{K}\varepsilon\end{aligned} \tag{34}$$

On the right-hand side of Eq. (34), one finds a term that survives at high Reynolds number. The model contains four free constants. In the standard procedure, these free constants of the model are determined on the basis of experimental data, existing literature and computer optimization. However, none of these exists for flow through the porous media.

3. Closure coefficients

The proposed κ–ε model has seven closure coefficients that have been introduced in replacing unknown double and triple correlations with expressions involving known turbulence and mean flow properties. The precise evaluation of these coefficients requires comparison of model predictions with measured experimental results. Unfortunately, experimental data of flow in porous media are not very numerous. Consequently, the best one can do is to set the values of the closure coefficients to assure agreement with the limiting case, namely the limit as $\phi\to1$ and $K\to\infty$, which leads to the clear flow case. There is no loss of generality in doing this, however, since these same general arguments have been used in the two-equation macroscopic turbulence model for incompressible flow in porous media [1]. Thus, in a first approximation, these coefficients can be assigned values close to the values for the clear flow. In summary, the closure coefficients for the second-order model proposed in this study are as follows:

$$\begin{aligned}&C_{\varepsilon1}=1.44,\quad C_{\varepsilon2}=1.92,\quad C_{\varepsilon3}=0.13,\\&C_\mu=0.08,\quad\text{and}\quad\sigma_k=1.0\end{aligned} \tag{35}$$

Values for the coefficients $C_{\varepsilon 4}$ and C_t must be determined from experiments or by performing direct numerical simulation.

4. Conclusion

The present work develop a modified form of the κ–ε closure scheme for turbulent flows of incompressible fluids in porous media. The proposed derivation of the transport equations is an improvement over that developed in Ref. [1] in the following areas:

1. The linearization of the Forchheimmer term assumed in Ref. [1] is relaxed, but in the present model, to approximate this term, the second-order term in the expansion is retained (Eq. (7)). It was then demonstrated that inclusion of the second-order term results in additional terms, which represent the form-drag effect of the porous matrix, in the ensemble-averaged momentum equation, the turbulent kinetic energy equation and the dissipation rate equation.
2. Comparing the turbulent kinetic energy equation, Eq. (21) and the dissipation rate equation, Eq. (34), with the corresponding equations, Eqs. (19) and (31) of Ref. [1], it is noticed that the model equations developed in this paper contain two additional closure coefficients: C_t in the turbulent kinetic energy equation and $C_{\varepsilon 4}$ in the dissipation rate equation. These additional closure coefficients allow one to use the clear flow values given in Eq. (34), because any departure from the clear flow may be adjusted via C_t and $C_{\varepsilon 4}$.

In order to explain and gain deeper insight into the closure scheme and the final transport equations, further comprehensive numerical, analytical and experimental work remains to be done. It is hoped that the work presented in this article will serve as an additional stimulant to initiate these activities and will lead to further research in the area of turbulence modeling for flows in porous media.

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